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The reals as full and balanced biframe

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ABSTRACT

This paper is the first part of a two-part investigation. It introduces full and balanced biframes which capture useful properties of the reals viewed as a biframe (or bitopological space). The subsequent paper will apply these concepts to the study of completions of quasi-nearness biframes.

We start with the smallest dense quotient for biframes. Next we discuss the reals as a biframe and introduce the key ideas of balanced, full and stable biframes. The crucial tool here is the frame pseudocomplement. We include a discussion of the relations between the newly introduced ideas and regularity. Order topology biframes are all regular, normal and balanced but not necessarily full. We consider the plane and various examples related to zero-dimensionality. We provide methods of transferring fullness and balancedness from domain to codomain and conversely under various kinds of maps.

Of particular importance to our later study of completions is the idea of a biframe map whose right adjoint preserves the first and second parts of the biframe. We give a result providing sufficient conditions for a map to have a part-preserving right adjoint. We present an example of a dense onto map (which is in fact a compactification) between normal, regular biframes whose right adjoint is not part-preserving. The paper concludes with internal properties of full and balanced biframes showing the particularly close connection between the first and second parts and ends with a final visit to the biframe of reals.

1. Introduction

As any topologist knows, the reals are exemplary. In the setting of bitopological spaces (bispaces) they are also of considerable importance. In this paper, we view the reals as a biframe, which means that we consider the usual topology on the real line generated by the lower and upper topologies. We isolate and explore three properties of this biframe for which we have introduced the names balanced, full and stable. We point out that these ideas are inherently biframe (or bispace) notions and have no counterpart in frames.

A biframe map automatically preserves the first and second parts; its right adjoint need not. In [19] the importance of biframe maps with part-preserving right adjoints was established in the study of the so-called perfect compactifications. In [9] they played an essential role in the construction of completions for quasi-nearness biframes. In this paper, we use the notions of balanced and full to establish sufficient conditions for a biframe map to have a part-preserving right adjoint.

In a subsequent paper [10] we will use these conditions to investigate biframes with quasi-uniform and quasi-nearness structures and their completions. In the light of this, one should view this paper as Part I and the subsequent paper as Part II of our exploration.

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We now turn to the structure of this paper. Section 2 lists largely familiar definitions and facts and introduces the smallest dense quotient for biframes. In Section 3 we discuss the reals as a biframe and introduce the key ideas of balanced, full and stable biframes. The crucial tool here is the frame pseudocomplement (which should not be confused with the biframe pseudocomplements of [22]). In Section 4 we discuss the relations between the newly introduced ideas and regularity. The bulk of this section consists of examples which show that all these notions are distinct. We also provide an example of a dense onto map (which is in fact a compactification) between normal, regular biframes whose right adjoint is not part-preserving. Section 5 considers order topologies on linearly ordered sets; the biframes in question are all regular, normal and balanced but not necessarily full. The section concludes with the plane and various examples related to zero-dimensionality. Section 6 provides methods of transferring fullness and balancedness from domain to codomain and conversely under various kinds of maps. The result providing sufficient conditions for a map to have a part-preserving right adjoint is presented here and the section concludes with an application of the transfer results to the smallest dense quotient. Section 7 concentrates on internal properties of full and balanced biframes showing the particularly close connection between the first and second parts and ends with a final visit to the biframe of reals.

2. Preliminaries

See [17,12,24] as references for frame theory. Most of the biframe notions in Definition 2.1 appear in the literature: see [1–3,7,8,20–22].

Definition 2.1. 1. (a) A *frame* L is a complete lattice in which the distributive law

$$x \wedge \bigvee\{y\colon\ y \in Y\} = \bigvee\{x \wedge y\colon\ y \in Y\}$$

holds for all $x \in L, Y \subseteq L$. A *frame map* is a set function between frames which preserves finite meets and arbitrary joins, and thus also the top (denoted 1) and the bottom (denoted 0) of the frame.

(b) If an element x of a frame L has a complement, that complement will be denoted by x'. The *pseudocomplement* of an element x is $x^* = \bigvee\{y \in L\colon\ y \wedge x = 0\}$.

2. (a) A *biframe* L is a triple $L = (L_0, L_1, L_2)$ in which L_0 is a frame, L_1 and L_2 are subframes of L_0, and $L_1 \cup L_2$ generates L_0. We call L_0 the *total part*, L_1 the *first part* and L_2 the *second part* of the biframe L.

(b) A *biframe map* $h : M \to L$ is a frame map from M_0 to L_0 such that the image of M_i under h is contained in L_i for $i = 1, 2$. We call the restriction $h|M_0$ the *total part* of the map h and $h|M_1 = h_1$ and $h|M_2 = h_2$ its *first* and *second parts* respectively.

(c) In what follows, in the context of biframes, we will reserve the subscript i for reference to the first and second parts only.

3. A biframe map $h : M \to L$ is *dense* if its total part is a dense frame map, i.e. $a = 0$ whenever $h(a) = 0$, for any $a \in M_0$.

4. A biframe map h is *onto* if its first and second parts are onto. (h is then onto on the total part.)

5. For a biframe map $h : M \to L$ we define the right adjoint of h as the right adjoint of its total part. Explicitly, if r is the right adjoint of h this means that $r(a) = \bigvee\{t \in M_0\colon\ h(t) \leqslant a\}$. If h is also onto, then $r(a) = \bigvee\{t \in M_0\colon\ h(t) = a\}$. We note that in general r is not a frame map, however it does preserve meets. There is also no *a priori* reason that r should map elements of L_i to elements of M_i. If $r[L_i] \subseteq M_i$ for $i = 1, 2$ we say that r is *part-preserving*.

6. For L a biframe and $x, y \in L_i$, we define $y \prec_i x$ to mean that there exists $c \in L_k$ $(k = 1, 2,\ k \neq i)$ such that $y \wedge c = 0$ and $x \vee c = 1$. We note that $x \prec_i x$ means that x has a complement, x' in L_0, and $x' \in L_k$, for $k \neq i$.

7. A biframe L is *regular* if each $x \in L_i$ $(i = 1, 2)$ can be expressed as a join $x = \bigvee\{y \in L_i\colon\ y \prec_i x\}$.

8. A biframe L is *normal* if, whenever $a \vee b = 1$ for $a \in L_1$ and $b \in L_2$, there exist $c \in L_2$ and $d \in L_1$ such that $c \wedge d = 0$, $a \vee c = 1$ and $b \vee d = 1$.

9. Let L be a biframe. For $x \in L_1$ write $x^\bullet = \bigvee\{t \in L_2\colon\ t \wedge x = 0\}$. Similarly, for $x \in L_2$, write $x^\bullet = \bigvee\{t \in L_1\colon\ t \wedge x = 0\}$. Then L is *de Morgan* if for each $x \in L_1 \cup L_2$, $x^\bullet \vee x^{\bullet\bullet} = 1$.

10. A biframe L is *zero-dimensional* if each $x \in L_i$ can be written: $x = \bigvee\{z \in L_i\colon\ z \prec_i z \leqslant x\}$ for $i = 1, 2$.

11. A biframe L is *strictly zero-dimensional* if each $x \in L_1$ has a complement x' in L_0 and $x' \in L_2$; and L_2 is generated by these complements. (A biframe would also be called strictly zero-dimensional if it satisfies this condition with L_1 and L_2 interchanged; but in this paper, we will not need the second version.)

12. A biframe L is *extremely zero-dimensional* if every $x \in L_1$ has a complement in L_2 and every $x \in L_2$ has a complement in L_1. (This property is called "Boolean" in [21,22].)

Definition 2.2. Recall that, for any frame L, its *Booleanization* is given by

$$L^{**} = \left\{x^{**}\colon\ x \in L\right\}$$

with meet as in L, but join given by

$$\bigsqcup X = \Bigl(\bigvee_L X\Bigr)^{**}.$$

The function $q_L : L \to L^{**}$ given by $q_L(x) = x^{**}$, is a frame map. It can be characterized as the *smallest dense quotient* of L, or as the unique dense quotient of L with a Boolean frame as codomain. (See [4].)

Definition 2.3. For any biframe M, the biframe M^{**} and the function $q_M : M \to M^{**}$ are defined as follows:

$$M^{**} = \left(M_0^{**}, M_1^{**}, M_2^{**}\right) \quad \text{where } M_j^{**} = \left\{x^{**}\colon x \in M_j\right\},\ j = 0, 1, 2,$$

$$q_M(x) = x^{**} \quad \text{for all } x \in M_0.$$

Then q_M is a dense, onto biframe map, referred to as the *smallest dense quotient* of M (which is justified by Lemma 2.4 below).

Lemma 2.4. *For a biframe M, the map $q_M : M \to M^{**}$ is the smallest dense quotient of M.*

Proof. We note that here the word "quotient" is used as a synonym for "onto map". The claim means that, for any dense, onto biframe map $f : M \to N$, there exists a biframe map $g : N \to M^{**}$ such that $gf = q_M$. Such a map g is necessarily unique, dense and onto. The proof follows directly from the corresponding result for frames. $\square$

Corollary 2.5. *For a biframe M, the map $q_M : M \to M^{**}$ is the unique dense quotient to a biframe with total part a Boolean frame.*

Proof. The map g provided by the proof of Lemma 2.4 would be a frame isomorphism and so also a biframe isomorphism. $\square$

Note 2.6. The right adjoint of $q_M : M \to M^{**}$ is the identical embedding, since, for any $x \in M_0$ and $a \in M_0^{**}$, $x^{**} \leqslant a \Leftrightarrow x \leqslant a$.

3. The reals as a biframe

The open sets of the real numbers, considered as a biframe (or bitopological space), are traditionally presented as follows: $\mathcal{R} = (\mathcal{O}\mathbb{R}, \mathcal{OD}\mathbb{R}, \mathcal{OU}\mathbb{R})$ where

- $\mathcal{O}\mathbb{R}$ consists of the usual open sets of the real line, $\mathbb{R}$, with join union and meet intersection.
- $\mathcal{OD}\mathbb{R} = \{(-\infty, a)\colon\ a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, the open downsets.
- $\mathcal{OU}\mathbb{R} = \{(b, \infty)\colon\ b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, the open upsets.

(See [14,16,18,11].) We will refer to $\mathcal{R}$ as *the biframe of reals*.

One of the aims of this paper is to isolate three properties that the biframe of reals has, and to investigate to what extent they are responsible for some exemplary real behavior. The properties in question are defined below; it is immediately clear that the biframe of reals satisfies them all.

Definition 3.1. Let $L = (L_0, L_1, L_2)$ be a biframe, and, for $a \in L_0$ let a^* be the pseudocomplement of a in L_0, that is, $a^* = \bigvee\{b \in L_0\colon\ b \wedge a = 0\}$.

1. L is called *balanced* if $x \in L_1$ implies that $x^* \in L_2$ and $x \in L_2$ implies that $x^* \in L_1$.
2. L is called *full* if $x \in L_1 \cup L_2$ implies that $x = x^{**}$.
3. L is called *double-pseudocomplement-stable* or just *stable* for short, if $x \in L_1$ implies that $x^{**} \in L_1$ and $x \in L_2$ implies that $x^{**} \in L_2$.

Clearly every balanced or full biframe is stable. The complete picture of how these properties relate and how they relate to regularity may be found in Proposition 4.2. That the biframe of reals is not unique in being balanced and full is illustrated by the next examples.

Example 3.2. Any extremely zero-dimensional biframe is balanced and full. As an instance, let X be any set, $p \in X$, $\mathcal{D}X$ the discrete topology on X, $\mathcal{E}X$ the p-exclusion topology on X and $\mathcal{I}X$ the p-inclusion topology on X; then $(\mathcal{D}X, \mathcal{E}X, \mathcal{I}X)$ is extremely zero-dimensional. (See [23, Examples 10 and 15], for the definitions of p-inclusion and p-exclusion topologies.)

Example 3.3. Let L_0 be the four element Boolean algebra, with a new top element added. Label the incomparable elements a and b and write $c = a \vee b$. Let $L_1 = \{0, a, 1\}$ and let $L_2 = \{0, b, 1\}$. Then $L = (L_0, L_1, L_2)$ is balanced and full, since $a^* = b$ and $b^* = a$. We note for later use that L is not regular, since $a \not\prec_1 a$. (On the other hand L is normal, since no non-trivial joins give 1.)

Example 3.4. If L is a Boolean frame, then (L, L, L) is of course full and balanced; and conversely, if (L, L, L) is full, then L is Boolean.

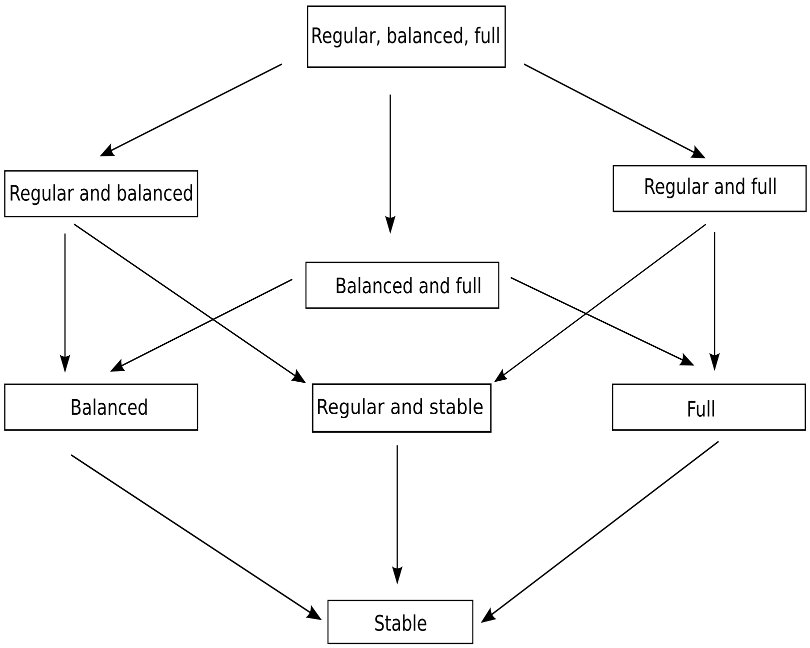
4. Relations between full, balanced, stable and regular biframes

In this section we investigate the relationships between balanced, full, stable and regular biframes. In Part II of this paper, we will consider quasi-nearness biframes and their completions, and all such biframes are indeed regular, so we are primarily interested in regular biframes here.

Lemma 4.1. *If L is a full biframe, then its total part, L_0, is a semiregular frame.*

Proof. Recall that a frame is semiregular if each of its elements is a join of regular elements. (An element x is regular if $x = x^{**}$.) Any $a \in L_0$ can be written $a = \bigvee_\alpha x_\alpha \wedge y_\alpha$, for some $x_\alpha \in L_1$ and $y_\alpha \in L_2$. But $x_\alpha = x_\alpha^{**}$ and $y_\alpha = y_\alpha^{**}$ for all α since L is full, so $a = \bigvee_\alpha x_\alpha^{**} \wedge y_\alpha^{**} = \bigvee_\alpha (x_\alpha \wedge y_\alpha)^{**}$. □

Proposition 4.2. *The relationships between balanced, full, stable and regular biframes are given in the diagram below, where arrows indicate implication and the absence of an arrow indicates the existence of a counterexample to the implication in question.*



Proof. All the implications are clear. Three counterexamples suffice for all the non-implications. Use (L, L, L) where L is regular but not Boolean as an example of a biframe that is regular and balanced, but not full. Use Example 4.3 for an example of a full and regular biframe that is not balanced and use Example 3.3 for an example of a biframe that is balanced and full but not regular. For an example of a biframe that is regular but not stable, see Example 4.6. □

Our next example introduces a particular biframe that will allow us (amongst other things) to construct an example of a dense onto biframe map that has right adjoint that is not part-preserving.

Example 4.3 *("Discrete–usual–discrete" or $\mathcal{DUD}$).* Let $\mathcal{D}\mathbb{R}$ be the discrete topology on the real line and $\mathcal{O}\mathbb{R}$ the usual topology on the real line. Define $\mathcal{DUD} = (\mathcal{D}\mathbb{R}, \mathcal{O}\mathbb{R}, \mathcal{D}\mathbb{R})$.

Claim. *$\mathcal{DUD}$ is regular, normal and full but not balanced.*

Regular: For $U \in \mathcal{O}\mathbb{R}$, we have $U \prec_1 U$ since $U' = \mathbb{R}\backslash U \in \mathcal{D}\mathbb{R}$. For $A \in \mathcal{D}\mathbb{R}$, we have $A = \bigcup\{\{a\} : a \in A\}$ and for such a, $\{a\} \prec_2 A$, since $\{a\}' \in \mathcal{O}\mathbb{R}$, $\{a\} \cap \{a\}' = \emptyset$ and $\{a\}' \cup A = \mathbb{R}$. In fact, since $\{a\} \prec_2 \{a\}$, it follows that $\mathcal{DUD}$ is zero-dimensional.

Normal: Suppose $U \cup A = \mathbb{R}$ for $U \in \mathcal{O}\mathbb{R}$ and $A \in \mathcal{D}\mathbb{R}$. Let $B = U'$ and $V = U$. Then $V \cap B = \emptyset$ and $V \cup A = \mathbb{R} = U \cup B$.

Full: Since $\mathcal{D}\mathbb{R}$ is a Boolean frame, $A = A^{**}$ for all $A \in \mathcal{D}\mathbb{R}$ including the case where $A \in \mathcal{O}\mathbb{R}$.

Not balanced: $(-\infty, 0) \in \mathcal{D}\mathbb{R}$ but $(-\infty, 0)^* = [0, \infty) \notin \mathcal{O}\mathbb{R}$.

In the next result, we consider the largest compactification of $\mathcal{DUD}$ (the biframe of Example 4.3). We will need the following results from [21]:

Let L be a normal regular biframe. Then

- $(\prec_1, \prec_2)$ is a strong inclusion on L, necessarily the largest.
- The largest compactification of L is given by the join map $\bigvee : \mathcal{R}L \to L$, where $\mathcal{R}L$ is described as follows. $(\mathcal{R}L)_i$ consists of those ideals J of L_0 such that

  1. J is generated by $J \cap L_i$, that is, for each $a \in J$, there exists $b \in J \cap L_i$ with $a \leqslant b$, and
  2. J is a regular ideal, that is, for each $c \in J \cap L_i$, there exists $d \in J \cap L_i$ with $c \prec_i d$.

  Then $(\mathcal{R}L)_0$ is the subframe of the frame of all ideals of L_0 that is generated by $(\mathcal{R}L)_1 \cup (\mathcal{R}L)_2$. (For details on the frame of ideals of a frame, see [12].)

Lemma 4.4. *In the biframe $\mathcal{DUD}$, $U \prec_1 V$ iff $U \subseteq V$ for $U, V \in \mathcal{O}\mathbb{R}$, and $A \prec_2 B$ iff $\overline{A} \subseteq B$ for $A, B \in \mathcal{D}\mathbb{R}$, where $\overline{A}$ denotes the closure of A in the usual topology on the reals.*

Proof. For $U \in \mathcal{O}\mathbb{R}$, we have $U \prec_1 U$ since $U' = \mathbb{R}\backslash U \in \mathcal{D}\mathbb{R}$ and $U \cup U' = \mathbb{R}, U \cap U' = \emptyset$. For $A, B \in \mathcal{D}\mathbb{R}$, if $A \prec_2 B$, there must exist a $V \in \mathcal{O}\mathbb{R}$ with $A \cap V = \emptyset$ and $V \cup B = \mathbb{R}$. Then $\overline{A}' \cup B = \mathbb{R}$ and so $\overline{A} \subseteq B$. Conversely, if $A, B \in \mathcal{D}\mathbb{R}$ with $\overline{A} \subseteq B$ then $V = \overline{A}' \in \mathcal{O}\mathbb{R}$ satisfies $A \cap V = \emptyset$ and $V \cup B = \mathbb{R}$. □

Proposition 4.5. *For the biframe $L = \mathcal{DUD}$, the largest compactification $\bigvee : \mathcal{R}L \to L$ of L has a right adjoint which is not part-preserving.*

Proof. We begin by describing explicitly the first and second parts of $\mathcal{R}L$ for this L.

An ideal J of L_0 is in $(\mathcal{R}L)_1$ if:

1. For all $A \in J$, there exists $U \in J \cap \mathcal{O}\mathbb{R}$ with $A \subseteq U$. (This is not vacuous.)
2. For all $U \in J \cap \mathcal{O}\mathbb{R}$, there exists $V \in J \cap \mathcal{O}\mathbb{R}$ with $U \prec_1 V$, that is, $U \subseteq V$. (This is vacuous.)

Note that for $U \in \mathcal{O}\mathbb{R}$, $\downarrow U = \{W \in \mathcal{D}\mathbb{R}\colon\ W \subseteq U\} \in (\mathcal{R}L)_1$.

An ideal J of L_0 is in $(\mathcal{R}L)_2$ if:

1. For all $A \in J$, there exists $B \in J \cap \mathcal{D}\mathbb{R}$ with $A \subseteq B$. (This is vacuous.)
2. For all $A \in J \cap \mathcal{D}\mathbb{R}$, there exists $B \in J \cap \mathcal{D}\mathbb{R}$ with $A \prec_2 B$, that is, $\overline{A} \subseteq B$. (This is not vacuous.)

So, if $J \in (\mathcal{R}L)_2$ then $A \in J$ iff $\overline{A} \in J$. So for A not closed in $\mathcal{O}\mathbb{R}$, $\downarrow A = \{W \in \mathcal{D}\mathbb{R}\colon\ W \subseteq A\} \notin (\mathcal{R}L)_2$.

We denote the right adjoint of the total part of the join map $\bigvee : \mathcal{R}L \to L$ by s. For $A \in \mathcal{D}\mathbb{R}$, the total part of L, $s(A) = \bigvee\{J \in (\mathcal{R}L)_0\colon\ \bigvee J \subseteq A\} = \bigvee\{J \in (\mathcal{R}L)_0\colon\ J \subseteq \downarrow A\}$.

For $U \in \mathcal{O}\mathbb{R}$, $s(U) = \downarrow U$ because $\downarrow U \in (\mathcal{R}L)_1$ and so $\downarrow U \in (\mathcal{R}L)_0$. Now take as instance $A = (0, 1)$. Then $A \in \mathcal{O}\mathbb{R}$ and $A \in \mathcal{D}\mathbb{R}$. From the above, $s(A) = \downarrow A$, but since A is not closed, $\downarrow A \notin (\mathcal{R}L)_2$. So s does not preserve the second part. □

Example 4.6 *("Sorgenfrey–usual–Sorgenfrey" or $\mathcal{SUS}$).* Let $\mathcal{S}\mathbb{R}$ be the Sorgenfrey topology on the real line, with base $\{[a, b)\colon a < b, a, b \in \mathbb{R}\}$, and $\mathcal{O}\mathbb{R}$ the usual topology on the real line. We define $\mathcal{SUS} = (\mathcal{S}\mathbb{R}, \mathcal{O}\mathbb{R}, \mathcal{S}\mathbb{R})$.

Claim. *$\mathcal{SUS}$ is regular but not stable.*

Regular: For $U \in \mathcal{O}\mathbb{R}$ we have that $U = \bigcup\{W \in \mathcal{O}\mathbb{R}\colon\ \overline{W} \subseteq U\}$ where $\overline{W}$ denotes the closure of W in the usual topology on the real line. For such W, $W \prec_1 U$ since $\overline{W}' \in \mathcal{S}\mathbb{R}$, $W \cap \overline{W}' = \emptyset$ and $\overline{W}' \cup U = \mathbb{R}$.

Now consider a basic open of $\mathcal{S}\mathbb{R}$, that is, a set of the form $[a, b)$. We have $[a, b) = \bigcup[a, b - \frac{1}{n})$ (taking only suitably large n) and $[a, b - \frac{1}{n}) \prec_2 [a, b)$ because $V = (-\infty, a) \cup (b - \frac{1}{2n}, \infty) \in \mathcal{O}\mathbb{R}$, with $[a, b - \frac{1}{n}) \cap V = \emptyset$ and $V \cup [a, b) = \mathbb{R}$.

Not stable: Take $U = (0, 1) \in \mathcal{O}\mathbb{R}$. Then $U^* = (-\infty, 0) \cup [1, \infty)$ (since the pseudocomplement is taken in $\mathcal{S}\mathbb{R}$). So $U^{**} = [0, 1) \notin \mathcal{O}\mathbb{R}$.

Note 4.7. The biframes in the last two examples were of the form (L_0, L_1, L_0). We note that such a biframe cannot be regular and balanced unless $L_1 = L_0$: Since L_0 would be regular (as the total part of a regular biframe is always regular) we must have $a = \bigvee\{(b^*)^*\colon\ b \in L_0, b^{**} \leqslant a\}$ (regular of course implies semiregular) for any $a \in L_0$. Since $b^* \in L_0$ (which we view as the second part) a balanced biframe would need b^{**} in L_1, but then $a \in L_1$ too.

Lemma 4.8. *Suppose that M is a regular biframe and $h : M \to L$ is a dense, onto biframe map with part-preserving right adjoint r. If $L_2 = L_0$ then $M_2 = M_0$.*

Proof. Take $a \in M_0$; then $a = \bigvee\{b \in M_0\colon b \prec a\}$ since M_0 is regular. Now $b \prec a$ implies that $b \leqslant rh(b) \leqslant a$. But $b \in M_0$ implies that $h(b) \in L_0$ which in turn implies that $h(b) \in L_2$ which finally implies that $rh(b) \in M_2$ since r was assumed to be part-preserving. So a is a join of elements of M_2 and is thus a member of M_2. $\square$

5. Order topologies and further examples

Example 5.1 *(Order Topology Biframes).* ([15,23]) We present here a collection of biframes that are all regular, normal and balanced; they need not be full, but we present a special case where they are full.

Let X be a set linearly ordered by $\leqslant$. We write $(-\infty, a) = \{x \in X\colon x < a\}$ and $(b, \infty) = \{x \in X\colon x > b\}$, as usual. Let L_0 be the topology on X with subbase $\{(-\infty, a)\colon a \in X\} \cup \{(b, \infty)\colon b \in X\}$. Let L_1 be the topology with base $\{(-\infty, a)\colon a \in X\}$ and let L_2 be the topology on X with base $\{(b, \infty)\colon b \in X\}$. We will call such (L_0, L_1, L_2) an *order topology biframe*.

Claim. *Any order topology biframe is regular, normal and balanced.*

Regular: First we note that if $U, V \in L_1$ with $U \subsetneqq V$ then $U \prec_1 V$: Take $x \in V\setminus U$. Then $(x, \infty) \in L_2$ and $U \cap (x, \infty) = \emptyset$, $(x, \infty) \cup V = X$. Now take $V \in L_1$. If $V = \bigcup\{U \in L_1\colon U \subsetneqq V\}$ then $V = \bigcup\{U \in L_1\colon U \prec_1 V\}$, as required. On the other hand, if $W = \bigcup\{U \in L_1\colon U \subsetneqq V\} \neq V$, we show that $V \prec_1 V$ which is sufficient to show regularity. (The case for $V \in L_2$ is similar.) To this end, take $x \in V\setminus W$ and $a \in W$. Then $a \in U$ for some $U \in L_1$ with $U \subsetneqq V$. Since $a \neq x$, $a < x$ or $x < a$. Then $a \in (-\infty, x)$ or $x \in U$. Since $x \notin U$ this shows that $a \in (-\infty, x)$. So $W \subseteq (-\infty, x) \subseteq V$. We note that $(-\infty, x) = V$ is impossible, since $x \in V$. So $W = (-\infty, x)$. We now claim that $V = (-\infty, x) \cup \{x\}$ which we write as $(-\infty, x]$. Certainly $(-\infty, x] \subseteq V$. Now suppose $z \in V$ and $z > x$: then $(-\infty, z) \subseteq W = (-\infty, x)$, so $z \leqslant x$, a contradiction. But $V = (-\infty, x]$ has a complement in L_0 which is (x, ∞) and $(x, \infty) \in L_2$. But then $V \prec_1 V$ as claimed.

Normal: Take $A \cup B = X$ for some $A \in L_1, B \in L_2$. We find $C \in L_2, D \in L_1$ with $C \cap D = \emptyset$ and $A \cup C = X = B \cup D$.

If $A \cap B = \emptyset$, the proof is complete, so assume that $A \cap B \neq \emptyset$.

Case 1: Suppose that A has a largest element, y. Then $y \in A \cap B$, since if $x \in A \cap B$ then $x \leqslant y$, so $y \in B$. Let $C = (y, \infty)$ and $D = A$. Then $C \cap D = \emptyset$ and $A \cup C = X = B \cup D$.

Case 2: If B has a smallest element one can use a similar argument to that of Case 1.

Case 3: Suppose that A has no largest element and B has no smallest element. Take $x \in A \cap B$ and set $C = (x, \infty)$, $D = (-\infty, x)$. Then $C \cap D = \emptyset$ and $A \cup C = X = B \cup D$.

Balanced: For $c, d \in X$ we write $(c, d) = \{x \in X\colon c < x < d\}$. Take $U \in L_1$: then $U = \bigcup_{a \in A}(-\infty, a)$ for some index set A. Now

$$U^* = \bigcup\bigl\{(c, d)\colon c, d \in X \text{ and } (c, d) \cap U = \emptyset\bigr\} \cup \bigl\{(s, \infty)\colon s \in X \text{ and } (s, \infty) \cap U = \emptyset\bigr\}.$$

We claim that actually $U^* = \bigcup\{(s, \infty)\colon s \in X \text{ and } (s, \infty) \cap U = \emptyset\}$ which exhibits U^* as a member of L_2 as needed. To justify the claim suppose that $(c, d) \cap U = \emptyset$; then it is easy to see that $(c, \infty) \cap U = \emptyset$ and we are done. The argument for $U \in L_2$ is similar.

We now give an example of an order topology biframe that is not full: Let $X = (-\infty, 0] \cup \{n \in \mathbb{Z}\colon n \geqslant 1\}$, with the usual order of the real line. Then $U = (-\infty, 0) \in L_1$ and $U^* = (0, \infty)$ and $U^{**} = (-\infty, 0] \neq U$.

We finally give a general property for order topology biframes that makes them full: Consider an order topology biframe L with the following properties:

1. If $A \in L_1$ and $A \neq X$, then A has no largest element.
2. If $B \in L_2$ and $B \neq X$, then B has no smallest element.

Such an order topology biframe is full. Take $U \in L_1$ and suppose that $U \neq U^{**}$. We note that $U^{**} \in L_1$, since L is balanced (and hence stable). Take $x \in U^{**}\setminus U$. This x is unique: if $x, y \in U^{**}\setminus U$ and $x < y$, then $y \in U^* \cap U^{**} = \emptyset$ which is a contradiction. This x is also the largest element of U^{**}, contradicting our hypothesis. The argument for $U \in L_2$ is similar.

Example 5.2 *(The plane as biframe).* Let $\mathcal{O}(\mathbb{R} \times \mathbb{R})$ denote the usual open sets of $\mathbb{R} \times \mathbb{R}$. For $(a, b) \in \mathbb{R} \times \mathbb{R}$ set

$$D_{(a,b)} = (-\infty, a) \times (-\infty, b),$$

$$U_{(a,b)} = (a, \infty) \times (b, \infty).$$

Let $\mathcal{OD}(\mathbb{R} \times \mathbb{R})$ have as base $\bigl\{D_{(a,b)}\colon (a, b) \in \mathbb{R} \times \mathbb{R}\bigr\}$.

Let $\mathcal{OU}(\mathbb{R} \times \mathbb{R})$ have as base $\bigl\{U_{(a,b)}\colon (a, b) \in \mathbb{R} \times \mathbb{R}\bigr\}$.

We define $\mathcal{R} \times \mathcal{R} = (\mathcal{O}(\mathbb{R} \times \mathbb{R}), \mathcal{OD}(\mathbb{R} \times \mathbb{R}), \mathcal{OU}(\mathbb{R} \times \mathbb{R}))$.

We note that $U \in \mathcal{OD}(\mathbb{R}\times\mathbb{R})$ iff U is open in the usual topology on $\mathbb{R}\times\mathbb{R}$, and $(x, y) \in U$, $s \leqslant x, t \leqslant y$ implies $(s,t) \in U$. A similar comment applies to $U \in \mathcal{OU}(\mathbb{R}\times\mathbb{R})$.

Claim. *$\mathcal{R}\times\mathcal{R}$ is regular, full and balanced.*

Regular: Note that, for any $\epsilon > 0$, $D_{(a-\epsilon,b-\epsilon)} \prec_1 D_{(a,b)}$ and $U_{(a+\epsilon,b+\epsilon)} \prec_2 U_{(a,b)}$. This gives regularity.

Full: Take $V \in \mathcal{OD}(\mathbb{R}\times\mathbb{R})$, and $(x, y) \notin V$. We show that $(x, y) \notin V^{**}$, that is, for any neighborhood W of (x, y), $W \cap V^* \neq \emptyset$. For W, take a basic neighborhood of the form $W = (x-\epsilon, x+\epsilon)\times(y-\epsilon, y+\epsilon)$, for some $\epsilon > 0$. Let $(s,t) = (x+\frac{1}{2}\epsilon, y+\frac{1}{2}\epsilon)$. Then $(s,t) \in U_{(x,y)}$ and $U_{(x,y)} \cap V = \emptyset$, so $(s,t) \in W \cap V^*$ as required. The argument for $V \in \mathcal{OU}(\mathbb{R}\times\mathbb{R})$ is similar.

Balanced: Take $V \in \mathcal{OD}(\mathbb{R}\times\mathbb{R})$. We show that V^* is the union of basic members of $\mathcal{OU}(\mathbb{R}\times\mathbb{R})$, so is in $\mathcal{OU}(\mathbb{R}\times\mathbb{R})$. Take $(x, y) \in V^*$. Since V^* is open in the usual topology of $\mathbb{R}\times\mathbb{R}$, there exists $\epsilon > 0$ such that $W = (x-\epsilon, x+\epsilon)\times(y-\epsilon, y+\epsilon) \subseteq V^*$. Let $(s,t) = (x-\frac{1}{2}\epsilon, y-\frac{1}{2}\epsilon)$. Then $U_{(s,t)} \subseteq V^*$ because $U_{(s,t)} \cap V = \emptyset$. [If $(a,b) \in U_{(s,t)} \cap V$, then $a > s, b > t$, so $(s,t) \in V$, but $(s,t) \in W \subseteq V^*$, a contradiction.] So $(x, y) \in U_{(s,t)} \subseteq V^*$, as required.

Note 5.3. We note that a straightforward proof shows that products of full (respectively balanced) biframes are in turn full (respectively balanced).

The last set of examples in this section explores relations between zero-dimensionality and full, balanced biframes.

Lemma 5.4. *A biframe is full, balanced and de Morgan iff it is extremely zero-dimensional.*

Proof. Suppose L is full, balanced and de Morgan. For $x \in L_1$, $x^\bullet = x^*$, since L is balanced. So $x^* \vee x^{**} = 1$. Since $x = x^{**}$ by fullness, we get $x \vee x^* = 1$ and $x^* \in L_2$ which makes L extremely zero-dimensional. The converse is clear. $\square$

Corollary 5.5. *A biframe is full, balanced and strictly zero-dimensional iff it is extremely zero-dimensional.*

Proof. This is because every strictly zero-dimensional biframe is de Morgan. (See [3].) $\square$

Example 5.6. We note that a full, balanced zero-dimensional biframe need not be extremely zero-dimensional: Let $\mathcal{O}\mathbb{Q}$ be the usual open sets of the rational line, and let

$$\mathcal{OD}\mathbb{Q} = \{(-\infty, a)\cap\mathbb{Q}\colon a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{Q}\},$$
$$\mathcal{OU}\mathbb{Q} = \{(b, \infty)\cap\mathbb{Q}\colon b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{Q}\}.$$

Then $(\mathcal{O}\mathbb{Q}, \mathcal{OD}\mathbb{Q}, \mathcal{OU}\mathbb{Q})$ is a case in point.

6. Transfer results

The following lemma provides a result that transfers the properties balanced, full and stable from the domain of a map to the codomain.

Lemma 6.1. *Let $h : M \to L$ be an onto biframe map that preserves pseudocomplements (that is, for which the total part $h : M_0 \to L_0$ is a frame map that preserves pseudocomplements). If M is balanced (respectively full, stable) then L is balanced (respectively full, stable).*

Proof. In each case we provide the argument for L_1 and the argument for L_2 is similar.

Balanced: Take $x \in L_1$: then there exists $a \in M_1$ with $h(a) = x$. Since $a^* \in M_2$, $h(a^*) \in L_2$. So $x^* = h(a)^* = h(a^*) \in L_2$. So L is balanced.

The arguments for full and stable are similar. $\square$

Corollary 6.2. (a) (*Dense quotients*) *If $h : M \to L$ is a dense, onto biframe map and M is balanced (respectively full, stable) then so is L.*

(b) (*Open quotients*) *If M is a biframe, $a \in M_0$ and $h : M \to {\downarrow}a$ is given by $h(x) = x \wedge a$, for all $x \in M_0$, and M is balanced (respectively full, stable) then ${\downarrow}a$ (viewed as a biframe) has the same property.*

Proof. (a) Dense, onto frame maps preserve pseudocomplements. (See [13,4].)

(b) We note that the biframe ${\downarrow}a$ has the form

$$\big(\{x \wedge a\colon x \in L_0\}, \{x \wedge a\colon x \in L_1\}, \{x \wedge a : \in L_2\}\big).$$

The result follows because open frame homomorphisms, of which this is a case, preserve pseudocomplements. (See [5,12].) □

Note 6.3. In [5], the authors consider frame maps satisfying $h(a^*) = h(a^{**})^*$. An argument similar to the above shows that if $h : M \to L$ is an onto biframe map whose total part satisfies this condition and M is balanced and full, then so is L.

The next lemma provides a transfer result for the properties balanced and stable from the codomain to the domain of a biframe map as long as it has a part-preserving right adjoint.

Lemma 6.4. *Let $h : M \to L$ be a dense, onto biframe map with part-preserving right adjoint, r. If L is balanced* (*respectively stable*) *then so is M.*

Proof. Again, we provide arguments that concern elements of M_1 in each case; the arguments for elements of M_2 are similar.

We first note that, for such h, and $a \in M_1$, we have $a^* = rh(a^*)$. (See Lemma 3.11 of [6].)

Balanced: For balanced L, take $a \in M_1$; then $h(a) \in L_1$, so $h(a)^* \in L_2$. But $h(a^*) = h(a)^*$, so $h(a^*) \in L_2$. Since r is part-preserving, $rh(a^*) \in M_2$, so $a^* \in M_2$. This makes M balanced.

Stable: The argument is similar; just use $h(a^{**}) = h(a)^{**}$. □

Note 6.5. We note that, under the conditions of Lemma 6.4 one cannot conclude from L being full that M is also full. As counterexample, take $q_M : M \to M^{**}$ (see Definition 2.3) where M is any biframe that is stable but not full. Then q_M is a dense, onto biframe map with part-preserving right adjoint (by Lemma 6.7), M^{**} is trivially full, because its total part is a Boolean frame, but M is not full.

The following result shows that knowledge of full and balanced biframes can indeed provide information about the existence of part-preserving right adjoints. This will be vital in Part II of this paper, when we use it to establish uniqueness of certain quasi-completions of quasi-nearness biframes.

Proposition 6.6. *Let M be a balanced biframe and L a full biframe. Let $h : M \to L$ be a dense, onto biframe map with right adjoint r. Then r is part-preserving.*

Proof. Take $a \in L_1$. Then $r(a) = \bigvee\{z \in M_0\colon\ h(z) = a\}$, since h is onto. Let $z \in M_0$ satisfy $h(z) = a$. Then $h(z) \wedge a^* = 0$. Now $a^* \in L_2$, since L is balanced (by Corollary 6.2). So $a^* = h(w)$ for some $w \in M_2$, since h is onto. Then $h(z) \wedge h(w) = 0$, and since h is dense, this gives $z \wedge w = 0$, so $z \leqslant w^*$. Further, $w^* \in M_1$, since M is balanced, and $h(w^*) = h(w)^*$, since dense, onto maps preserve pseudocomplements. Thus $h(w^*) = a^{**} = a$, since L is full. So $z \leqslant w^* \leqslant r(a)$, with $w^* \in M_1$. So $r(a)$ can be expressed as a join of elements of M_1, so $r(a) \in M_1$. The argument for $a \in L_2$ is similar. □

We now list a few straightforward facts relating the smallest dense quotient to the properties balanced, full and stable. The importance of these will become apparent when we consider the so-called "smooth" quasi-nearness structures in Part II of this paper.

Lemma 6.7. *Let M be a biframe.*

(a) *M is balanced iff M^{**} is balanced and M is stable.*
(b) *M is full iff $M^{**} = (M_0^{**}, M_1, M_2)$ and $q_M|_{M_i}$ is the identity map for $i = 1, 2$.*
(c) *M is stable iff q_M has part-preserving right adjoint.*

Proof. (a) Apply Corollary 6.2 to deduce that M^{**} is balanced (and any balanced biframe is stable). Conversely, take $x \in M_1$. Then $x^{**} \in M_1^{**}$, so, since x^* is the complement of x^{**} in M_0^{**}, we have $x^* \in M_2^{**}$. So there exists a $y \in M_2$ with $x^* = y^{**}$. Since M is stable, $y^{**} \in M_2$, so $x^* \in M_2$, as required.

(b) and (c) are clear. (See Note 2.6.) □

7. Internal properties

In this section, we consider internal properties of full and balanced biframes, and in particular how their first and second parts relate to each other.

Proposition 7.1. *A biframe is full and balanced iff the functions $\alpha : L_1 \to L_2$ and $\beta : L_2 \to L_1$, both given by taking pseudocomplements in L_0, are* 1-1 *and onto.* [*Note that here α and β are merely set functions, not frame maps*; *in particular they are order-reversing.*]

Proof. Suppose L is full and balanced. Since L is balanced, α and β are functions with the claimed domains and codomains. If $x^* = y^*$ for some $x, y \in L_1$, then $x = x^{**} = y^{**} = y$ since L is full, and α is thus 1–1. For $z \in L_2$, $z^* \in L_1$ since L is balanced. Also $\alpha(z^*) = z^{**} = z$, since L is full. So α is onto. The arguments for β are similar.

Conversely, suppose that α and β are both 1–1 and onto. Then automatically L is balanced. For fullness, take $x \in L_1$. Then $x^{**} \in L_1$ too. Since $x^* = (x^{**})^*$ and α is 1–1, it follows that $x = x^{**}$. The argument for $x \in L_2$ is similar. $\square$

Corollary 7.2. *A biframe L is full and balanced iff $L_1^* = L_2$ and $L_2^* = L_1$, where $L_i^* = \{x^*\colon x \in L_i\}$, $i = 1, 2$.*

Note 7.3. It follows from the above that, if L is a full and balanced biframe, L_2 is uniquely determined by L_1 and L_0, since $L_2 = L_1^*$.

Proposition 7.4. *If L is a full and balanced biframe, then L_1 and L_2 are each closed under the (arbitrary) meet in L_0.*

Proof. Recall that, for any subset Y of a frame, $(\bigvee Y)^* = \bigwedge\{y^*\colon\ y \in Y\}$. Take $X \subseteq L_1$. Since L is full, $x = x^{**}$ for all $x \in X$. Now

$$\bigwedge X = \bigwedge_{x\in X} x^{**} = \left(\bigvee_{x\in X} x^*\right)^*$$

where all meets are taken in L_0. Since L is balanced, $x^* \in L_2$ for each $x \in X$, and hence $\bigvee_{x\in X} x^* \in L_2$. Thus, in the light of L being balanced, $\bigwedge X \in L_1$, because it is the pseudocomplement of a member of L_2.

The argument for L_2 is similar. $\square$

It is of course entirely possible for an element to be in the first part as well as the second part of a biframe. In the case of full and balanced biframes, there is a constraint involved, as the next lemma shows.

Lemma 7.5. *Let L be a full and balanced biframe. If $x \in L_1$ and $x \in L_2$ then x is complemented (in L_0).*

Proof. Suppose that L is full and balanced and $x \in L_1 \cap L_2$. Then $x^* \in L_1 \cap L_2$. Now $(x \vee x^*)^{**} = (x^* \wedge x^{**})^* = 0^* = 1$. Since $x \vee x^* \in L_1$, we must have $x \vee x^* = (x \vee x^*)^{**} = 1$. $\square$

The next example shows that a frame can be presented as the total part of a full and balanced biframe in more than one way.

Example 7.6. Let $\mathcal{D}\mathbb{Z}$ be the discrete topology on the integers. The following are non-isomorphic full and balanced biframes with $\mathcal{D}\mathbb{Z}$ as total part:

- $(\mathcal{D}\mathbb{Z}, \mathcal{D}\mathbb{Z}, \mathcal{D}\mathbb{Z})$.
- The order topology biframe on $\mathbb{Z}$ with its usual order.
- $(\mathcal{D}\mathbb{Z}, \mathcal{E}\mathbb{Z}, \mathcal{I}\mathbb{Z})$ where $\mathcal{E}\mathbb{Z}$ is the 0-exclusion topology and $\mathcal{I}\mathbb{Z}$ is the 0-inclusion topology.

We finally present two lemmas concerning the biframe of reals which shows how significant the open downsets and upsets are.

Lemma 7.7. *If $(\mathcal{O}\mathbb{R}, K_1, K_2)$ is a full and balanced biframe and $K_1 \supseteq \mathcal{OD}\mathbb{R}$, then $K_1 = \mathcal{OD}\mathbb{R}$.*

Proof. Suppose $U \in \mathcal{O}\mathbb{R}$ and that $U \in K_1$ but $U \notin \mathcal{OD}\mathbb{R}$. Consider the case $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$ a countable union of disjoint open intervals. Such a U is a regular open set since it is a member of K_1. Consider any $j \in \omega$; then $(-\infty, a_j) \in K_1$ and $U \in K_1$, so $(-\infty, a_j) \cup U \in K_1$, but this subset is not a regular open subset, which is a contradiction of the fact that K_1 is a collection of regular open subsets closed under joins (unions). Other cases for U can be treated similarly. $\square$

Lemma 7.8. *If $(\mathcal{O}\mathbb{R}, K_1, K_2)$ is a full and balanced biframe and $K_1 \subseteq \mathcal{OD}\mathbb{R}$, then $K_1 = \mathcal{OD}\mathbb{R}$.*

Proof. Suppose that $a \in \mathbb{R}$ and $(-\infty, a) \notin K_1$. Note that $K_1 \subseteq \mathcal{OD}\mathbb{R}$ implies that $K_2 \subseteq \mathcal{OU}\mathbb{R}$ by Corollary 7.2. Since $(\mathcal{O}\mathbb{R}, K_1, K_2)$ is a biframe, $(-\infty, a)$ can be written in the form $(-\infty, a) = \bigcup_\alpha (-\infty, c_\alpha) \cap (d_\alpha, \infty)$, where $(-\infty, c_\alpha) \in K_1$ and $(d_\alpha, \infty) \in K_2$ for all α. Without loss of generality, we may assume that $(-\infty, c_\alpha) \cap (d_\alpha, \infty) \neq \emptyset$ for each α. But then $(-\infty, a) = \bigcup_\alpha (-\infty, c_\alpha) \in K_1$ which is a contradiction. $\square$

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